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LETTER TO THE EDITOR

Perturbation-induced radiative losses in collision of NSE solitons

Yuri S Kivshar and Boris A Malomed†

Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, Kharkov 310164, USSR

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Abstract. The total energy (and its spectral density) emitted during the collision of two solitons described by the conservatively perturbed non-linear Schrödinger equation (NSE) is calculated by means of the perturbation theory based on the inverse scattering transform. It is shown that the total emitted energy (at large relative velocity) does not depend on the relative internal phase of the colliding solitons. The same results are obtained for solitons in a system of two weakly coupled non-linear Schrödinger equations.

The non-linear Schrödinger equation (NSE)

$$iu_t + u_{xx} + 2|u|^2 u = \varepsilon P[u] \tag{1}$$

ε being a small parameter, has important applications in numerous physical problems (see, e.g., Pésceli 1985, Kodama 1985). In the case $\varepsilon = 0$ (NSE proper) this equation is well known to be exactly integrable by means of the inverse scattering transform (IST) (Zakharov *et al* 1980). The simplest exact localised solution is the soliton:

$$u_s(x, t) = 2i\eta \frac{\exp[-2i\xi x - 4i(\xi^2 - \eta^2)t - i\phi]}{\cosh[2\eta(x + 4\xi t - x_0)]} \tag{2}$$

where η is the amplitude, -4ξ is the velocity and ϕ and x_0 are initial phase parameters of this soliton. In the absence of perturbations ($\varepsilon = 0$) collisions between solitons are pure elastic, i.e. they are not accompanied by radiative energy losses. Besides, in the exactly integrable case, 'many-particle' collisions reduce to a superposition of 'two-particle' collisions.

In real physical problems there commonly occur different small perturbations $P[u]$ breaking the exact integrability. The effect of a dissipative perturbation is rather evident: it damps a NSE soliton (Kaup and Newell 1978, Karpman 1979, Bullough *et al* 1982). Conservative perturbations result in more subtle effects. One soliton persists under the action of these perturbations. However, collisions between the solitons are no longer elastic and many-particle effects in collisions arise as well. The latter effects have recently been considered for kinks of the sine-Gordon equation (Kivshar and Malomed 1986). In the present letter we aim to investigate radiative losses accompanying collisions between two NSE solitons in the presence of conservative perturbations. We shall calculate the spectral density of the energy emitted during the collision and the total energy losses. We also calculate the collision-induced losses of the charge

† Permanent address: P P Shirshov Institute of Oceanology, Moscow 117218, USSR.

(or 'number of particles', see Zakharov *et al* (1980)), i.e. one more elementary integral of motion of the unperturbed NSE. We reveal that in the limit of large relative velocity ($\xi \to \infty$) the emitted energy does not depend on the phase differences $\Delta\phi = \phi_1 - \phi_2$ of the two colliding solitons, ϕ_1 and ϕ_2 being their phase parameters defined in (2).

First of all we consider the perturbation

$$P[u] = |u|^4 u. \tag{3}$$

This perturbation is universal: it describes the first term of the expansion of dispersion in powers of an amplitude which breaks the exact integrability (see, e.g., Zakharov *et al* 1980).

In terms of IST the radiation part of the wave field governed by the NSE is described by the continuous spectrum. The scattering data pertaining a continuous spectrum are determined by the complex coefficient $b(\lambda, t)$, λ being the real spectral parameter (e.g. Zakharov *et al* 1980, Ablowitz and Segur 1981). Since the radiation is generated by a small perturbation due to the overlapping of colliding solitons, the coefficient $b(\lambda, t)$ is small: $|b(\lambda, t)|^2 \ll 1$, and the radiation energy spectral density is (see, e.g., Zakharov *et al* 1980)

$$\mathscr{E}(\lambda, t) \approx \frac{4}{\pi}\lambda^2 |b(\lambda, t)|^2. \tag{4}$$

In the absence of perturbations the continuous spectrum is decoupled from the solitons, i.e. $\mathrm{d}|b(\lambda, t)|^2/\mathrm{d}t = 0$. Under the action of the perturbation (3) $b(\lambda, t)$ evolves according to the equation (Kaup and Newell 1978, Karpman 1979)

$$\frac{\partial b(\lambda, t)}{\partial t} = 4\mathrm{i}\lambda^2 b(\lambda, t) - \varepsilon \int_{-\infty}^{\infty} \mathrm{d}x\, |u(x, t)|^4 \{u(x, t)\Psi_2^{*2}(x, t; \lambda) + u^*(x, t)\Psi_1^{*2}(x, t; \lambda)\}. \tag{5}$$

Here $\Psi_{1,2}(x, t; \lambda)$ are the so-called Jost functions (see, e.g., Zakharov *et al* 1980, Ablowitz and Segur 1981). We shall consider only the case when the amplitudes η (see (2)) of the two colliding solitons are equal. Moreover, the calculation can be accomplished explicitly for the case of large velocities $\pm 4\xi$ of the colliding solitons.

So, assuming the soliton velocities $\pm 4\xi$ in the centre-of-mass reference frame to be large, $\xi \gg \eta$, it can be verified (see, e.g., Malomed 1985) that the two-soliton Jost functions of discrete spectrum and the wave potential can be split into 'one-particle' potentials presented, for example, in the paper of Karpman (1979). Therefore the Jost functions of a continuous spectrum are simplified too.

Inserting the Jost functions into (5) and assuming the radiation is originally absent, i.e. taking the initial condition for (5) in the form

$$b(\lambda, t = -\infty) = 0$$

(recall that one soliton is an exact solution of the NSE with conservative perturbation (3) and generates no emission), we can calculate the total spectral density of the energy emitted during the collision as $(4/\pi)\lambda^2 |b(\lambda, t = +\infty)|^2$ (see (4)). Omitting the details of a rather lengthy calculation, we shall display the result: the emitted energy spectral density is ($\xi \gg \eta$)

$$\mathscr{E}(\lambda) = \frac{4\varepsilon^2\eta^6\lambda^2}{\pi\xi^2}\left\{\left|F\left(\frac{\lambda-\xi}{\eta}\right)\right|^2 + \left|F\left(-\frac{\lambda+\xi}{\eta}\right)\right|^2 + 2\,\mathrm{Re}\left[\exp(\mathrm{i}\Delta\phi)\left(G\left(\frac{\lambda-\xi}{\eta}\right) + G\left(-\frac{\lambda+\xi}{\eta}\right)\right)\right]\right\} \tag{6}$$

where $F(z)$ and $G(z)$ are the complex functions:

$$F(z) \approx (\pi/15) Q(z)(z-\mathrm{i})^{-2} \operatorname{sech}(\pi z/2) \tag{7}$$

$$G(z) \approx (2\pi^2/9)(\xi/\eta)^5 \exp(-\pi\xi/\eta) Q(z)(z^2+1)^{-2} \operatorname{sech}(\pi z/2). \tag{8}$$

Here $Q(z)$ is the complex polynomial:

$$Q(z) = 49z^4 + 156z^2 + 45 - 4\mathrm{i}z(z^2+9). \tag{9}$$

The parameter $\Delta\phi = \phi_1 - \phi_2$ in (6) is the phase difference between the two solitons at the moment of collision. As is seen from (6) and (7)-(9), the radiation spectral density maxima lie at the spectrum points $\lambda = \pm\xi$, the maximum value itself being

$$\mathscr{E}_{\max} \sim \varepsilon^2 \eta^6. \tag{10}$$

The radiation energy is concentrated in the spectral range of width $\sim\eta$ in the vicinity of the points $\lambda = \pm\lambda_m$ where $\lambda_m = \xi$. Outside those ranges the spectral density falls exponentially as $\exp[-(\pi/\eta)|\lambda_m \pm \lambda|]$. Note that (6) is symmetric relative to changing the sign of λ which evidently reflects the symmetry between the left and right directions in the present problem. Note also that expression (6) immediately determines the density of the emitted energy in terms of the radiation wavenumbers k, since λ and k are related trivially: $k = 2\lambda$ (e.g. Zakharov *et al* 1980).

The term in square brackets in (6), which depends on $\Delta\phi$, is exponentially small compared with the first two which are independent of $\Delta\phi$. In (6) we have neglected the terms containing faster dependencies on $\Delta\phi$ since those terms are exponentially small in comparison with those taken into account. As we see, we arrive here at a rather curious inference: the emitted energy (at $\xi \to \infty$) does not depend on the parameter $\Delta\phi$. Note that an analogous conclusion has recently been obtained (Kivshar and Malomed 1986) for the perturbation-induced radiative energy loss accompanying the collision of two sine-Gordon solitons (kinks): the emitted energy in the limit of large relative velocity is asymptotically independent on the relative polarity σ of the two kinks (the 'Pomeranchuk theorem'). The inference obtained in the present problem is even more strong: σ for the sine-Gordon kinks is the sign parameter assuming only two values ± 1, while here $\Delta\phi$ is the continuous parameter.

The total emitted energy E_{em} is determined by the obvious formula:

$$E_{\mathrm{em}} = \int_{-\infty}^{\infty} \mathscr{E}(\lambda)\,\mathrm{d}\lambda = A\varepsilon^2\eta^7 + \varepsilon^2\xi^5\eta^2 \exp(-\pi\xi/\eta)(B_1 \cos\Delta\phi + B_2 \sin\Delta\phi). \tag{11}$$

Here A, B_1 and B_2 are numerical constants determined by integral representations. Their approximate values are

$$A \approx 690.5 \qquad B_1 \approx 1200.5 \qquad B_2 \approx 173.3.$$

It is interesting to note that, as one sees from (11), E_{em} does not vanish at $\xi \to \infty$. In this respect the NSE differs from the sine-Gordon equation, where the total emitted energy falls to zero with the growth of the relative velocity.

Besides energy, the NSE possesses two more elementary integrals of motion: momentum $P = \mathrm{i}\int \mathrm{d}x\,(u_x u^* - u_x^* u)$ and charge $N = \int \mathrm{d}x\,|u|^2$ ('number of particles'). The spectral density of emitted momentum $\mathscr{P}(\lambda)$ is simply related to $\mathscr{E}(\lambda)$ (Zakharov *et al* 1980): $\mathscr{P}(\lambda) = -(2\lambda)^{-1}\mathscr{E}(\lambda)$, but the emitted charge spectral density is $\mathscr{N}(\lambda) = (2\lambda)^{-2}\mathscr{E}(\lambda)$. The total emitted momentum is, evidently, zero. Finally, the total emitted charge is

$$N_{\mathrm{em}} \approx E_{\mathrm{em}}/4\xi^2.$$

Now we consider a system of two linearly coupled NSE:

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + 2|u|^2 u = \varepsilon v_{xx} \\ &\mathrm{i}v_t + v_{xx} + 2|v|^2 v = \varepsilon u_{xx}. \end{aligned} \tag{12}$$

The system (12) is a natural generalisation of (1) with the perturbed Hamiltonian

$$H_{\text{pert}} = -\varepsilon \int_{-\infty}^{\infty} \mathrm{d}x (u_x v_x^* + u_x^* v_x).$$

It is, for example, a simple model describing the interactions of small-amplitude non-linear excitations (small-amplitude breathers) in a system of two long weakly interacting Josephson junctions (see Mineev *et al* 1981). We consider the collision of solitons belonging to the u and v subsystems of (12) and deal with radiative effects. It should be noted that these problems are formulated similarly to those considered within the framework of one conservatively perturbed NSE (1) and (3). However, for the two-component system (12) these problems prove to be essentially simple and make it possible to obtain clearer and more detailed results. For instance, in the case of one NSE the spectral density of the radiation energy emitted by colliding solitons can be explicitly calculated only in the limit of fast solitons (see (6)–(9) and (11)). In the case of the system (12) we may perform the calculations for arbitrary values of the relative velocity.

In the framework of the so-called adiabatic approximation (see Karpman 1979) the collision between the u soliton and v soliton is pure elastic, i.e. it results only on the perturbation-induced phase shifts:

$$\begin{aligned} &(\Delta x_0)_1 = (\Delta x_0)_2 = 0 \\ &\Delta\phi_1 = \Delta\phi_2 = -\frac{\pi^3 \varepsilon}{8\eta^2} \xi^2 \cos(\Delta\phi) \frac{\tanh(\pi\xi/2\eta)}{\cosh^2(\pi\xi/2\eta)}. \end{aligned} \tag{13}$$

In the next (radiative) approximation the interactions of these solitons are accompanied by radiative energy losses. The radiation energy spectral densities for u and v subsystems may be calculated with the help of (4) and may be presented as follows:

$$\mathscr{E}_1(\lambda; \xi, \eta) = \frac{\pi^2 \varepsilon^2 \lambda^2 \eta^2}{4\xi^2[(\lambda+\xi)^2 + \eta^2]} [A^2 + B^2 + 2AB\cos(\Delta\phi)]$$

$$\mathscr{E}_2(\lambda; \xi, \eta) = \mathscr{E}_1(-\lambda; \xi, \eta) = \mathscr{E}_1(\lambda; -\xi, \eta).$$

Here A and B are the real functions:

$$A = \frac{ah(b)}{\sinh(\pi a/2)\cosh(\pi b/2)} \qquad B = \frac{f(c)h(-d)}{\sinh(\pi c/2)\cosh(\pi d/2)}$$

and $h(x)$ and $f(x)$ are the simple polynomials:

$$h(x) = x^2 - 2(\xi/\eta)x + 1 \qquad f(x) = \eta^2 x + 2\eta(\lambda+\xi).$$

The functions a, b, c and d are connected with the spectral parameter λ as follows:

$$\begin{aligned} a &= (\lambda^2 - 2\lambda\xi - 7\xi^2 + \eta^2)/4\eta\xi \\ b &= (\lambda^2 + 2\lambda\xi + 5\xi^2 + \eta^2)/4\eta\xi \\ c &= (\lambda^2 - 2\lambda\xi + \xi^2 + \eta^2)/4\eta\xi \\ d &= (\lambda^2 + 2\lambda\xi - 3\xi^2 + \eta^2)/4\eta\xi. \end{aligned}$$

However, the total emitted energies for u and v subsystems may be calculated explicitly only for large relative velocity. In the limit $\xi \gg \eta$ the total emitted energy $(E_{em})_1$ for the u subsystem and those $(E_{em})_2$ for the v subsystem are determined by the simple formula ($\xi \gg \eta$)

$$(E_{em})_1 = (E_{em})_2 \approx 98.62\varepsilon^2 \eta \xi^2. \tag{14}$$

The next (non-written) term in (14), which depends on $\Delta\phi$, is exponentially small on η/ξ as compared with the one taken into account in (14). So we see, as above, that the emitted energy asymptotically (at the large relative velocity of solitons) does not depend on the relative phase parameter $\Delta\phi$.

In conclusion we would like to note that the radiative effects investigated in the present letter (see, e.g., our results (6)-(9) and (11)) can be, in principle, observed in experiments, e.g. as the emission of electromagnetic waves generated by collisions between Langmuir solitons in plasma (see, e.g., Pésceli 1985), or as the emission of dispersive waves induced by the overlapping of solitons in non-linear optical fibres (see, e.g., Kodama 1985).

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